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THE GAUGED NONLINEAR SCHRÖDINGER EQUATION ON THE PLANE: A REGULARIZED MODEL WITH VORTEX SOLUTIONS

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We consider a complex scalar field interacting with the Chern — Simons gauge field in 2+1 dimensions. In the relativistic case this model is known to support vortexlike solutions. However, its nonrelativistic counterpart was found not to possess even the condensate (i.e. spatially uniform) solution. Here we propose a new self-consistent nonrelativistic Chern — Simons gauge theory which does have both the condensate solution and a variety of vortex- and bubblelike solitons. When the scalar self-interaction is ϕ^4, the solutions satisfy self-duality equations.

The investigation has been carried out at the Laboratory of Computing Techniques and Automation, JINR, University of Cape Town and University of Natal (Durban).

Нелинейное уравнение Шредингера с калибровочным полем на плоскости: регуляризованная модель с вихревыми решениями

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Рассматривается комплексное скалярное поле, взаимодействующее с калибровочным полем Черна — Саймонса в (2+1)-мерном пространстве. Известно, что, хотя в релятивистском случае эта модель допускает вихревые решения, ее нерелятивистский вариант не описывает даже конденсат, т.е. пространственно-однородное состояние. В настоящей заметке предлагается новая самосогласованная нерелятивистская модель с членом Черна — Саймонса, обладающая как конденсатным решением, так и набором топологических и нетопологических солитонов. В случае взаимодействия ϕ^4, решения удовлетворяют уравнениям самодуальности.

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Introduction. Recently there has been much interest in (2+1)-dimensional gauge theories with the Chern-Simons term. The Chern-Simons theories were proposed in the context of condensed matter physics to describe low-energy phenomena in quasiplanar systems of anyons, particles with fractional statistics. These phenomena include e. g. the high-T_c superconductivity and the fractional quantum Hall effect.

The role of the anyonlike objects is played by vortices, topologically nontrivial two-dimensional localized structures. Accordingly vortex solutions have been in the focus of mathematical studies of the Chern-Simons theories. Most progress, so far, has been made in the relativistic case, when the matter field satisfies the nonlinear Klein-Gordon equation. As far as the nonrelativistic limit is concerned (which is of course more attractive for applications), only nontopological, "bell-like" solitons were found [1]. Furthermore, it can readily be shown [2] that even the *condensate* (i. e. nonzero constant solution) is not admitted by the nonrelativistic model introduced in [1].

In the present note we demonstrate that this drawback originates in the fact that the gauged nonlinear Schrödinger equation in its standard form [1] is not suitable for describing *dark solitons*, i.e. solitons with the nonvanishing background, such as vortices and bubbles. Accordingly, we propose a new version of the model which is completely compatible with the nonvanishing ("condensate") boundary conditions. As its predecessor, the new model is self-consistent: the conserved matter current serves as a source of the gauge field.

Inadequacy of the standard model. The failure of the standard nonrelativistic model [1] to possess even the condensate solution can be traced back to its Lagrangian formulation. This drawback is inherited from the nongauged version of the model.

Indeed, consider the one-dimensional nonlinear Schrödinger equation with a general nonlinearity,

$$i\phi_t + \phi_{xx} + F(\rho)\phi = 0. \qquad (1)$$

Here $\rho = |\phi|^2$, and $F(\rho) = -dU/d\rho$. The standard Lagrangian for eq. (1),

$$\mathcal{L} = \frac{i}{2}(\phi_t\phi^* - \phi_t^*\phi) - |\phi_x|^2 - U(\rho), \qquad (2)$$

does not *automatically* produce correct integrals of motion for solutions with nonvanishing boundary conditions. First of all, the number of particles integral,

$$N = \int \left(\frac{\partial \mathcal{L}}{\partial \phi_t} i\phi - \frac{\partial \mathcal{L}}{\partial \phi_t^*} i\phi^* \right) dx, \tag{3}$$

corresponding to the global $U(1)$ invariance of the system, takes the form $N = \int \rho \, dx$. This integral, of course, diverges for solutions with $|\phi|^2$ approaching ρ_0 at infinity. The regularized number of particles,

$$N = \int (\rho - \rho_0) \, dx, \tag{4}$$

is obtained by the *ad hoc* subtraction of the background contribution.

The standard definition of momentum,

$$P = \int \left(\phi_x \frac{\partial \mathcal{L}}{\partial \phi_t} + \phi_x^* \frac{\partial \mathcal{L}}{\partial \phi_t^*} \right) dx, \tag{5}$$

does not yield the correct expression either. For the Lagrangian (2) one obtains $P = \frac{i}{2} \int (\phi \phi_x^* - \phi^* \phi_x) \, dx$. It appears that this definition is not compatible with the Hamiltonian structure of the model [4]. Indeed, varying P gives

$$\delta P = i \int (\phi_x^* \delta\phi - \phi_x \delta\phi^*) dx + \rho_0 \delta \mathrm{Arg}\, \phi \big|_{-\infty}^{+\infty}.$$

The appearance of the boundary term here makes it impossible to find the functional derivatives $\delta P/\delta\phi$ and $\delta P/\delta\phi^*$. Consequently, the Poisson bracket of P with some other functional, e. g. Hamiltonian, cannot be evaluated. The only definition compatible with the Hamiltonian structure of the model is*

$$\begin{aligned} P &= \frac{i}{2} \int (\phi \phi_x^* - \phi^* \phi_x) \, dx - \rho_0 \mathrm{Arg}\, \phi \big|_{-\infty}^{+\infty} \\ &= \int (\phi_x^* \phi - \phi_x \phi^*) \left(1 - \frac{\rho_0}{\rho} \right) dx. \end{aligned} \tag{6}$$

*It is appropriate to note that it was this modified definition of momentum that permitted the formulation of a stability criterion for moving dark solitons [3], [4].

Proceeding to two dimensions, the standard definition

$$P = \frac{i}{2} \int (\phi \nabla \phi^* - \phi^* \nabla \phi) \, d^2 r \tag{7}$$

is even less suitable, since in general the integral (7) diverges.

Now the gauged nonlinear Schrödinger equation reads

$$i\phi_t - eA_0\phi + \mathbf{D}^2\phi + F(\rho)\phi = 0, \tag{8}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and the Abelian gauge field A_μ satisfies

$$\kappa \partial_\beta F^{\beta\alpha} + \frac{\mu}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} = eJ^\alpha. \tag{9}$$

Here $J^\mu = (J_0, \mathbf{J})$ designates the conserved matter current:

$$J_0 = \rho = |\phi|^2, \tag{10}$$

$$\mathbf{J} = \frac{1}{i}\{\phi^*(\mathbf{D}\phi) - \phi(\mathbf{D}\phi)^*\}, \tag{11}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, Greek and Latin indices run over 0,1,2 and 1,2, respectively. The parameter e is a gauge coupling, and κ and μ control the relative contributions of the Maxwell and Chern-Simons terms in the corresponding Lagrangian:

$$\mathcal{L} = \frac{i}{2}(\phi^*(D_0\phi) - \phi(D_0\phi)^*) - (D_k\phi)^*(D_k\phi) - \frac{\kappa}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{4}\epsilon^{\mu\alpha\beta} A_\mu F_{\alpha\beta} - U(\rho). \tag{12}$$

This system inherits the drawback of its nongauged counterpart. The momentum, defined by

$$P_i = \int \left[\frac{\partial \phi}{\partial x^i} \frac{\partial \mathcal{L}}{\partial \phi_t} + \frac{\partial \phi^*}{\partial x^i} \frac{\partial \mathcal{L}}{\partial \phi_t^*} + \frac{\partial A^\alpha}{\partial x^i} \frac{\partial \mathcal{L}}{\partial A_t^\alpha} \right] d^2 r, \tag{13}$$

equals

$$P_i = \int \left[\frac{i}{2}(\phi D_i \phi^* - \phi^* D_i \phi) - \kappa \epsilon_{ij} E^j B \right] d^2 r = \int \rho \, (\partial_i \mathrm{Arg}\, \phi - eA^i) \, d^2 r - \kappa \int \epsilon_{ij} E^j B \, d^2 r. \tag{14}$$

Here $\mathbf{E} = -\nabla A_0$ is electric and $B = -F_{12}$ magnetic fields. The momentum in this case is finite, provided $A^i \to \frac{1}{e}\partial_i \mathrm{Arg}\, \phi$ at infinity. However, the functional derivatives $\delta P_i / \delta \phi$ and $\delta P_i / \delta \phi^*$ still cannot be calculated.

The new model. Thus, a natural question is: Is it possible to find a Lagrangian producing the same Euler-Lagrange equations as (2) but at the same time yielding the correct integrals of motion? We claim that the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\phi_t\phi^* - \phi_t^*\phi)\left(1 - \frac{\rho_0}{\rho}\right) - |\phi_x|^2 - U(\rho), \tag{15}$$

satisfies both these requirements. It produces the same NLS (1), while formulas (3) and (5) yield the correct integrals N and P, as given exactly by (4) and (6).

We propose to adopt the Lagrangian (15) as the basis for the nonrelativistic gauge theory. In two dimensions, and after the introduction of the Chern-Simons – Maxwell gauge field, it takes the following form:

$$\mathcal{L} = \frac{i}{2}(\phi^*(D_0\phi) - \phi(D_0\phi)^*)\left(1 - \frac{\rho_0}{\rho}\right) - (D_k\phi)^*(D_k\phi) - \frac{\kappa}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{4}\epsilon^{\mu\alpha\beta}A_\mu F_{\alpha\beta} - U(\rho). \tag{16}$$

As expected, this Lagrangian produces the correct number of particles,

$$N = \int (\rho - \rho_0)\, d^2r, \tag{17}$$

and momentum compatible with the Hamiltonian structure of the model:

$$P_i = \int \left[\frac{i}{2}(\phi D_i\phi^* - \phi^* D_i\phi)\left(1 - \frac{\rho_0}{\rho}\right) - \kappa\epsilon_{ij}E^jB\right] d^2r = \int (\rho - \rho_0)\left[\partial_i \mathrm{Arg}\,\phi - eA^i\right] d^2r + \kappa\int \epsilon_{ij}E^jB\, d^2r. \tag{18}$$

The field equations resulting from (16) however differ from those of the standard model (12). The difference lies in the definition of the density of the number of particles [the charge component of the vector $J^\mu = (J_0, \mathbf{J})$]:

$$J_0 = \rho - \rho_0. \tag{19}$$

Other than that, the field equations are the same, eqs. (8), (9), with the spatial part of the current being given by the standard expression (11). The Hamiltonian of the modified system has the standard form:

$$H = \int \left[|\mathbf{D}\phi|^2 + U(\rho) + \frac{\kappa}{2} (\mathbf{E}^2 + B^2) \right] d^2 r. \qquad (20)$$

The difference in the definition of J_0, however, drastically changes the properties of the model. It turns out that the modified model not only possesses the condensate solution but also exhibits a self-dual limit and a rich variety of vortex and bubble-like solitons.

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